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**On exact solutions of the Dirac equation
in a homogeneous magnetic field in the Lobachevsky space**

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There are constructed exact solutions of the quantum-mechanical Dirac equation for a spin $S=1/2$ particle in Riemannian space of constant negative curvature, hyperbolic Lobachevsky space, in presence of an external magnetic field, analogue of the homogeneous magnetic field in the Minkowski space. A generalized formula for energy levels, describing quantization of the motion of the particle in magnetic field on the background of the Lobachevsky space geometry, has been obtained.

1. Introduction

The quantization of a quantum-mechanical particle in the homogeneous magnetic field belongs to classical problems in physics [1, 2, 3]. In [4, 5, 6], exact solutions for a scalar particle in extended problem, particle in external magnetic field on the background of Lobachevsky H_3 and Riemann S_3 spatial geometries were found. A corresponding system in the frames of classical mechanics was examined in [7, 8, 9]. In the present paper, we consider a similar problem for a particle with spin $1/2$ described by Dirac equation in Lobachevsky space in presence of the external magnetic field.

2. Cylindric coordinates ant the Dirac equation in hyperbolic space H_3

In the Lobachevsky space, let us use an extended cylindric coordinates

$$dS^2 = dt^2 - \text{ch}^2 z (dr^2 + \text{sh}^2 r d\phi^2) - dz^2, \quad (1)$$

$$u^1 = \text{ch } z \text{ sh } r \cos \phi, \quad u^2 = \text{ch } z \text{ sh } r \sin \phi, \quad u^3 = \text{sh } z, \quad u^0 = \cosh z \text{ ch } r;$$

where $x^j = (r, \phi, z)$: $r \in [0, +\infty)$, $\phi \in [0, 2\pi]$, $z \in (-\infty, +\infty)$; the curvature radius ρ is taken as a unit of the length. An analogue of usual homogeneous magnetic field is defined as

$$A_\phi = -2B \text{sh}^2 \frac{r}{2} = -B (\cosh r - 1). \quad (2)$$

To coordinates (1) there corresponds the tetrad

$$e_{(a)}^\beta = \begin{vmatrix} 1 & 0 & 0 & 0 \\ 0 & \cosh^{-1} z & 0 & 0 \\ 0 & 0 & \cosh^{-1} z \sinh^{-1} r & 0 \\ 0 & 0 & 0 & 1 \end{vmatrix}. \quad (3)$$

Christoffel symbols Γ_{jk}^r and Ricci rotation coefficients γ_{abc} are

$$\Gamma_{jk}^r = \begin{vmatrix} 0 & 0 & \text{th } z \\ 0 & -\text{sh } r \cosh r & 0 \\ \text{th } z & 0 & 0 \end{vmatrix}, \quad \Gamma_{jk}^\phi = \begin{vmatrix} 0 & \text{cth } r & 0 \\ \text{cth } r & 0 & \text{th } z \\ 0 & \text{th } z & 0 \end{vmatrix},$$

$$\Gamma_{jk}^z = \begin{vmatrix} -\cosh z \text{ sh } z & 0 & 0 \\ 0 & -\text{sh } z \cosh z \text{ sh}^2 r & 0 \\ 0 & 0 & 0 \end{vmatrix},$$

$$\gamma_{122} = \frac{1}{\cosh z \tanh r}, \quad \gamma_{311} = \tanh z, \quad \gamma_{322} = \tanh z.$$

A general covariant Dirac equation (for more detail see [10]) takes the form

$$\left[i\gamma^0 \partial_t + \frac{i\gamma^1}{\text{ch } z} (\partial_r + \frac{1}{2} \frac{1}{\text{th } r}) + \gamma^2 \frac{i\partial_\phi + eB(\text{ch } r - 1)}{\text{ch } z \text{ sh } r} + i\gamma^3 (\partial_z + \text{th } z) - M \right] \Psi = 0. \quad (4)$$

With the substitution $\Psi = \varphi / \sqrt{\text{sh } r \text{ ch } z}$ eq. (4) becomes simpler

$$\left[i\gamma^1 \frac{\partial}{\partial r} + \gamma^2 \frac{i\partial_\phi + eB(\text{ch } r - 1)}{\text{sh } r} + \text{ch } z \left(i\gamma^0 \frac{\partial}{\partial t} + i\gamma^3 \frac{\partial}{\partial z} - M \right) \right] \varphi = 0. \quad (5)$$

Solutions of this equation will be searched in the form

$$\varphi = e^{-i\epsilon t} e^{im\phi} \begin{vmatrix} f_1(r, z) \\ f_2(r, z) \\ f_3(r, z) \\ f_4(r, z) \end{vmatrix},$$

so that

$$\left[i\gamma^1 \frac{\partial}{\partial r} - \gamma^2 \frac{m - eB(\text{ch } r - 1)}{\text{sh } r} + \text{ch } z \left(\epsilon\gamma^0 + i\gamma^3 \frac{\partial}{\partial z} - M \right) \right] \begin{vmatrix} f_1(r, z) \\ f_2(r, z) \\ f_3(r, z) \\ f_4(r, z) \end{vmatrix} = 0. \quad (6)$$

Taking the Dirac matrices in spinor basis, we get radial equations for $f_a(t, z)$

$$\begin{aligned} \left(\frac{\partial}{\partial r} + \mu \right) f_4 + \text{ch } z \frac{\partial f_3}{\partial z} + i \text{ch } z (\epsilon f_3 - M f_1) &= 0, \\ \left(\frac{\partial}{\partial r} - \mu \right) f_3 - \text{ch } z \frac{\partial f_4}{\partial z} + i \text{ch } z (\epsilon f_4 - M f_2) &= 0, \\ \left(\frac{\partial}{\partial r} + \mu \right) f_2 + \text{ch } z \frac{\partial f_1}{\partial z} - i \text{ch } z (\epsilon f_1 - M f_3) &= 0, \\ \left(\frac{\partial}{\partial r} - \mu \right) f_1 - \text{ch } z \frac{\partial f_2}{\partial z} - i \text{ch } z (\epsilon f_2 - M f_4) &= 0, \end{aligned} \quad (7)$$

where $\mu(r) = [m - eB(\text{ch } r - 1)] / \text{sh } r$. With linear restriction

$$f_3 = A f_1, \quad f_4 = A f_2 \quad (8)$$

eqs. (7) give

$$\begin{aligned} \left(\frac{\partial}{\partial r} + \mu \right) f_2 + \text{ch } z \frac{\partial f_1}{\partial z} + i \text{ch } z \left(\epsilon - \frac{M}{A} \right) f_1 &= 0, \\ \left(\frac{\partial}{\partial r} + \mu \right) f_2 + \text{ch } z \frac{\partial f_1}{\partial z} + i \text{ch } z (-\epsilon + MA) f_1 &= 0, \\ \left(\frac{\partial}{\partial r} - \mu \right) f_1 - \text{ch } z \frac{\partial f_2}{\partial z} + i \text{ch } z \left(\epsilon - \frac{M}{A} \right) f_2 &= 0, \\ \left(\frac{\partial}{\partial r} - \mu \right) f_1 - \text{ch } z \frac{\partial f_2}{\partial z} + i \text{ch } z (-\epsilon + MA) f_2 &= 0. \end{aligned} \quad (9)$$

The system (9) is self-consistent only if

$$\epsilon - \frac{M}{A} = -\epsilon + MA \quad \implies \quad A = A_{1,2} = \frac{\epsilon \pm p}{M}, \quad p = \sqrt{\epsilon^2 - M^2}. \quad (10)$$

So, the problem is reduced to the following systems

$$\begin{aligned} \left(\frac{\partial}{\partial r} + \mu\right) f_2 + \text{ch } z \frac{\partial f_1}{\partial z} + i \text{ch } z (-\epsilon + MA) f_1 &= 0, \\ \left(\frac{\partial}{\partial r} - \mu\right) f_1 - \text{ch } z \frac{\partial f_2}{\partial z} + i \text{ch } z (-\epsilon + MA) f_2 &= 0. \end{aligned} \quad (11)$$

Thus, we have two different (but similar) cases

$$AM = \epsilon + p,$$

$$\begin{aligned} \left(\frac{\partial}{\partial r} + \mu\right) f_2 + \text{ch } z \left(\frac{\partial}{\partial z} + ip\right) f_1 &= 0, \\ \left(\frac{\partial}{\partial r} - \mu\right) f_1 - \text{ch } z \left(\frac{\partial}{\partial z} - ip\right) f_2 &= 0; \end{aligned} \quad (12)$$

$$AM = \epsilon - p,$$

$$\begin{aligned} \left(\frac{\partial}{\partial r} + \mu\right) f_2 + \text{ch } z \left(\frac{\partial}{\partial z} - ip\right) f_1 &= 0, \\ \left(\frac{\partial}{\partial r} - \mu\right) f_1 - \text{ch } z \left(\frac{\partial}{\partial z} + ip\right) f_2 &= 0. \end{aligned} \quad (13)$$

For definiteness, let us consider the system (12) (transition to the case (13) is performed by the formal change $p \implies -p$). Let us search solutions in the form

$$f_1 = Z_1(z) R_1(r), \quad f_2 = Z_2(z) R_2(r). \quad (14)$$

Eqs. (12) result in

$$\begin{aligned} \left(\frac{\partial}{\partial r} + \mu\right) Z_2 R_2 + \text{ch } z \left(\frac{\partial}{\partial z} + ip\right) Z_1 R_1 &= 0, \\ \left(\frac{\partial}{\partial r} - \mu\right) Z_1 R_1 - \text{ch } z \left(\frac{\partial}{\partial z} - ip\right) Z_2 R_2 &= 0. \end{aligned} \quad (15)$$

Introducing the separating constant λ :

$$\text{ch } z \left(\frac{\partial}{\partial z} + ip\right) Z_1 = \lambda Z_2, \quad \text{ch } z \left(\frac{\partial}{\partial z} - ip\right) Z_2 = \lambda Z_1 \quad (16)$$

we arrive at the radial system

$$\left(\frac{\partial}{\partial r} + \mu\right) R_2 + \lambda R_1 = 0, \quad \left(\frac{\partial}{\partial r} - \mu\right) R_1 - \lambda R_2 = 0. \quad (17)$$

3. Solution of the equation in z -variable

From (16) it follows a second order differential equation for $Z_1(z)$:

$$\frac{d^2 Z_1}{dz^2} + \frac{\text{sh } z}{\text{ch } z} \frac{dZ_1}{dz} + \left(p^2 + ip \frac{\text{sh } z}{\text{ch } z} - \frac{\lambda^2}{\text{ch}^2 z}\right) Z_1 = 0. \quad (18)$$

In a new variable $y = (1 + \text{th } z)/2$, eq. (18) will give

$$\left[4y(1-y) \frac{d}{dy} + 2(1-2y) \frac{d}{dy} + p^2 \left(\frac{1}{1-y} + \frac{1}{y} \right) + ip \left(\frac{1}{1-y} - \frac{1}{y} \right) - 4\lambda^2 \right] Z_1 = 0. \quad (19)$$

With the substitution $Z_1 = y^A(1-y)^B Z(y)$, eq. (19) leads to

$$4y(1-y) \frac{d^2 Z}{dy^2} + 4 \left[2A + \frac{1}{2} - (2A + 2B + 1)y \right] \frac{dZ}{dy} + \left[\frac{2A(2A-1) + p(p-i)}{y} + \frac{2B(2B-1) + p(p+i)}{1-y} - 4(A+B)^2 - 4\lambda^2 \right] Z = 0.$$

Requiring

$$A = -\frac{ip}{2}, \quad \frac{1+ip}{2}, \quad B = \frac{ip}{2}, \quad \frac{1-ip}{2}; \quad (20)$$

the equation for Z_1 is reduced to that of hypergeometric type

$$y(1-y) \frac{d^2 Z}{dz^2} + \left[2A + \frac{1}{2} - (2A + 2B + 1)y \right] \frac{dZ}{dz} - [(A+B)^2 + \lambda^2] Z = 0 \quad (21)$$

with parameters given by

$$\alpha = +i\lambda + A + B, \quad \beta = -i\lambda + A + B, \quad \gamma = 2A + \frac{1}{2},$$

$$Z_1 = \left(\frac{e^z}{\cosh z} \right)^A \left(\frac{e^{-z}}{\cosh z} \right)^B F(\alpha, \beta, \gamma; \frac{e^z}{2 \cosh z}). \quad (22)$$

There arise four possibility depending on the choice of A, B in (20), they provide us with solutions of different behavior in the region $z \rightarrow \pm\infty$.

$$\begin{aligned} \text{Variant 1,} \quad A = \frac{1+ip}{2}, \quad B = \frac{1-ip}{2}, \quad A+B=1, \quad A-B=ip, \\ \alpha = i\lambda + 1, \quad \beta = -i\lambda + 1, \quad \gamma = ip + \frac{3}{2}, \quad Z_1 = \frac{e^{ipz}}{\cosh z} F(\alpha, \beta, \gamma; \frac{e^z}{2 \cosh z}), \\ z \rightarrow +\infty, \quad Z_1 \rightarrow \frac{e^{ipz}}{e^z} = 0, \quad z \rightarrow -\infty, \quad Z_1 \rightarrow \frac{e^{ipz}}{e^{-z}} = 0. \end{aligned} \quad (23)$$

$$\begin{aligned} \text{Variant 2,} \quad A = -\frac{ip}{2}, \quad B = \frac{ip}{2}, \quad A+B=0, \quad A-B=-ip, \\ \alpha = i\lambda, \quad \beta = -i\lambda, \quad \gamma = -ip + \frac{1}{2}, \quad Z_1 = e^{-ipz} F(\alpha, \beta, \gamma; \frac{e^z}{2 \cosh z}). \end{aligned} \quad (24)$$

$$\begin{aligned} \text{Variant 3,} \quad A = \frac{1+ip}{2}, \quad B = \frac{ip}{2}, \quad A+B=ip+1/2, \quad A-B=1/2, \\ \alpha = +i\lambda + \frac{1}{2} + ip, \quad \beta = -i\lambda + \frac{1}{2} + ip, \quad \gamma = ip + \frac{3}{2}, \\ Z_1 = \frac{e^{z/2}}{(\cosh z)^{ip+1/2}} F(\alpha, \beta, \gamma; \frac{e^z}{2 \cosh z}), \\ z \rightarrow +\infty, \quad Z_1 \rightarrow e^{-ipz}, \quad z \rightarrow -\infty, \quad Z_1 \rightarrow e^{+ipz} e^{-\infty} = 0. \end{aligned} \quad (25)$$

Variant 4, $A = -\frac{ip}{2}$, $B = \frac{1-ip}{2}$, $A + B = -ip + 1/2$, $A - B = -1/2$,
 $\alpha = +i\lambda + \frac{1}{2} - ip$, $\beta = -i\lambda + \frac{1}{2} - ip$, $\gamma = -ip + \frac{1}{2}$,
 $Z_1 = \frac{e^{-z/2}}{(\operatorname{ch} z)^{-ip+1/2}} F(\alpha, \beta, \gamma; \frac{e^z}{2 \cosh z})$,
 $z \rightarrow +\infty$, $Z_1 \rightarrow e^{+ip} z e^{-z} = 0$, $z \rightarrow -\infty$, $Z_1 \rightarrow e^{-ip} z$. (26)

4. Solution of the equations in r -variable

From radial equations (17) it follows a second order equation for R_1 :

$$\left(\frac{d^2}{dr^2} - \frac{d\mu}{dr} - \mu^2 + \lambda^2 \right) R_1 = 0. \quad (27)$$

Remembering on the meaning of $\mu(r)$ (for shortness let us note eB as B) we obtain explicit form of the equation for R_1 :

$$\frac{d^2 R_1}{dr^2} + \left[\frac{m \operatorname{ch} r + B(\operatorname{ch} r - 1)}{\operatorname{sh}^2 r} - \frac{[m - B(\operatorname{ch} r - 1)]^2}{\operatorname{sh}^2 r} + \lambda^2 \right] R_1 = 0. \quad (28)$$

With the variable $y = (1 + \operatorname{ch} r)/2$. eq. (28) gives

$$y(1-y) \frac{d^2 R_1}{dy^2} + \left(\frac{1}{2} - y \right) \frac{dR_1}{dy} - \left[\lambda^2 + \frac{m^2}{4} \left(\frac{1}{y} + \frac{1}{1-y} \right) + \frac{m}{4} \left(\frac{1}{y} - \frac{1}{1-y} \right) + \frac{mB}{y} - B^2 \left(1 - \frac{1}{y} \right) + \frac{B}{2y} \right] R_1 = 0. \quad (29)$$

Making the substitution $R_1 = y^A(1-y)^C R(y)$, eq. (29) is reduced to

$$y(1-y) \frac{d^2 R}{dy^2} + \left[2A + \frac{1}{2} - (2A + 2C + 1)y \right] \frac{dR}{dy} + \left[\frac{A^2 - A/2 - m^2/4 - m/4 - mB - B^2 - B/2}{y} + \frac{C^2 - C/2 - m^2/4 + m/4}{1-y} - (A+C)^2 - \lambda^2 + B^2 \right] R = 0. \quad (30)$$

Requiring

$$A = -\frac{2B+m}{2}, \quad \frac{2B+m+1}{2}, \quad C = \frac{m}{2}, \quad \frac{1-m}{2}. \quad (31)$$

we arrive at an equation of hypergeometric type

$$y(1-y) \frac{d^2 R}{dy^2} + \left[2A + \frac{1}{2} - (2A + 2C + 1)y \right] \frac{dR}{dy} - \left[(A+C)^2 + \lambda^2 - B^2 \right] R = 0,$$

so that

$$\alpha = A + C + \sqrt{B^2 - \lambda^2}, \quad \beta = A + C - \sqrt{B^2 - \lambda^2}, \quad \gamma = 2A + \frac{1}{2},$$

$$R_1 = (1 + \cosh r)^A (1 - \cosh r)^C F(\alpha, \beta, \gamma; \frac{1 + \cosh r}{2}). \quad (32)$$

To have wave solutions finite in the origin $r = 0$ (corresponding geometrical points belong to the axis z : $u_0 = \cosh z$, $u_3 = \sinh z$, $u_1 = 0$, $u_2 = 0$) and in infinity $r \rightarrow \infty$, we must take positive C and negative A , such that $C + A < 0$:

$$R_1 = (1 + \cosh r)^A (1 - \cosh r)^C F(\alpha, \beta, \gamma; \frac{1 + \cosh r}{2}); \quad C \geq 0, \quad A < 0. \quad (33)$$

Let us follow all four possibilities to choose the parameters

1. $C = \frac{m}{2} \geq 0, \quad A = \frac{2B + m + 1}{2} < 0, \quad C + A = B + m + \frac{1}{2} < 0;$
2. $C = \frac{1 - m}{2} \geq 0, \quad A = \frac{2B + m + 1}{2} < 0, \quad C + A = B + 1 < 0;$
3. $C = \frac{m}{2} \geq 0, \quad A = -\frac{2B + m}{2} < 0, \quad C + A = -B < 0;$
4. $C = \frac{1 - m}{2} \geq 0, \quad A = -\frac{2B + m}{2} < 0, \quad C + A = -B - m + \frac{1}{2} < 0.$

Therefore, only two variants are appropriate

$$3. \quad m > 0; \quad 4. \quad -2B < m \leq 1. \quad (34)$$

Respective expressions for radial functions are

$$\text{Variant 3, } m > 0, \quad C = m/2, \quad A = -B - m/2 < 0,$$

$$R_1 = (1 + \cosh r)^{-B - m/2} (1 - \cosh r)^{m/2} F(\alpha, \beta, \gamma; \frac{1 + \cosh r}{2}),$$

$$\alpha = -B + \sqrt{B^2 - \lambda^2}, \quad \beta = -B - \sqrt{B^2 - \lambda^2}, \quad \gamma = -2B - m + \frac{1}{2}; \quad (35)$$

with the quantization rule

$$\alpha = -n \implies \sqrt{B^2 - \lambda^2} = B - n \implies \lambda^2 = +2Bn - n^2. \quad (36)$$

To have radial function finite at the infinity $r \rightarrow \infty$, the following inequality must be imposed

$$A + C + n < 0 \implies B - n > 0; \quad (37)$$

which insures the positive square root $\sqrt{B^2 - \lambda^2}$ in (36).

$$\text{Variant 4, } -2B < m \leq 1, \quad C = 1/2 - m/2,$$

$$A = -B - m/2 < 0, \quad C + A = -B - m + 1/2 < 0,$$

$$R_1 = (1 + \cosh r)^{-B - m/2} (1 - \cosh r)^{1/2 - m/2} F(\alpha, \beta, \gamma; \frac{1 + \cosh r}{2}),$$

$$\alpha = (-B - m + 1/2) + \sqrt{B^2 - \lambda^2},$$

$$\beta = (-B - m + 1/2) - \sqrt{B^2 - \lambda^2}, \quad \gamma = -2B - m + \frac{1}{2}. \quad (38)$$

The quantization rule is

$$\begin{aligned} \alpha = -n & \implies \sqrt{B^2 - \lambda^2} = B + m - 1/2 - n \\ & \implies \lambda^2 = +2B(n - m + \frac{1}{2}) - (n - m + \frac{1}{2})^2. \end{aligned} \quad (39)$$

The inequality must be fulfilled

$$A + C + n < 0 \implies B + m - 1/2 - n > 0, \quad (40)$$

which make positive the root $\sqrt{B^2 - \lambda^2}$ in (39).

Thus, the energy spectrum for spin 1/2 particle in the magnetic field in the Lobachevsky space model is given by two formulae

$$\begin{aligned} 3. \quad & n < B, \quad m > 0, \quad \lambda^2 = +2Bn - n^2; \\ 4. \quad & n < B + m - 1/2, \quad -2B < m \leq 1, \\ & \lambda^2 = +2B(n - m + \frac{1}{2}) - (n - m + \frac{1}{2})^2. \end{aligned} \quad (41)$$

The transition to the limit os the flat Minkowski space is realized in accordance with the rules

$$\begin{aligned} & \lambda^2 \rightarrow \frac{P_z^2 \rho^2}{\hbar^2} = \lambda_0^2 \rho^2, \quad B \rightarrow \frac{eB}{\hbar c} \rho^2 \\ 2. \quad & \lambda_0^2 = \frac{2eB}{\hbar} n; \quad 4. \quad \lambda_0^2 = \frac{2eB}{\hbar} (n - m + \frac{1}{2}). \end{aligned} \quad (42)$$

In the end there should be given a clarifying remarks. In fact, the above used relationship $-i\partial_\phi \Psi = m \Psi$ represents transformed from cartesian coordinates to cylindrical an eigen-value equation for the third projection of the the total angular momentum of the Dirac particle

$$\hat{J}_3 \Psi_{Cart} = (-i\frac{\partial}{\partial \phi} + \Sigma_3) \Psi_{Cart} = m \Psi = m \Psi_{Cart}; \quad (43)$$

this means that for the quantum number m are permitted half-integer values $m = \pm\frac{1}{2}, \pm\frac{3}{2}, \dots$

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